Method of Undetermined coefficients.

Used to find a particular solution

Example: $y''-2y'-3y=3$

Try $Y=A$. $\Rightarrow A=-1 \Rightarrow Y=-1$

Gen. sol'n: $y=C_1e^{3t}+C_2e^{-t}-1$

Example: $y''-y'-2y=t^2+1$

Try $Y=At^2+Bt+C \Rightarrow A=-\frac{1}{2},\ B=\frac{1}{2},\ C=-\frac{5}{4}$

Gen. sol'n: $y=C_1e^{2t}+C_2e^{-t}-\frac{1}{2}t^2+\frac{1}{2}t-\frac{5}{4}$.

Rmk: If $g(t)$ is a polynomial up to degree n, only need to try n-th degree.

Rmk: If $g(t)=t^n$, the Y should still be set to be a generic n-th degree polynomial.

Example: $y''-y-2y=t^3$.

$Y=At^3+Bt^2+Ct+D$ Start from the degree of $g(t)$

$Y'=3At^2+2Bt+C$ Write all the way down to the

$Y''=6At+2B$ Constant term!

$$Y''-Y'-2Y = 6At+2B-3At^2-2Bt-C-2At^3-2Bt^2-2Ct-2D$$

$$= -2At^3+(-3A-2B)t^2+(6A-2B-2C)t+2B-C-2D$$

Set it equal to t^3

$\Rightarrow -2A=1,\ -3A-2B=0,\ 6A-2B-2C=0,\ 2B-C-2D=0$

$\Rightarrow A=-\frac{1}{2},\ B=-\frac{3A}{2}=+\frac{3}{4},\ C=B-3A=\frac{3}{4}+\frac{3}{2}=\frac{9}{4}$

$D=\frac{2B-C}{2}=\frac{3}{4}-\frac{9}{8}=-\frac{3}{8}$

Particular sol'n: $Y=-\frac{1}{2}t^3+\frac{3}{4}t^2+\frac{9}{4}t-\frac{3}{8}$

Gen. sol'n: $y=C_1e^{2t}+C_2e^{-t}-\frac{1}{2}t^3+\frac{3}{4}t^2+\frac{9}{4}t-\frac{3}{8}$.

Example: $y''-y'-2y=e^{3t}$

Idea: RHS is exponential function. Try exp. funcs.

$Y=Ae^{3t}$ $\quad Y'=3Ae^{3t}$, $Y''=9Ae^{3t}$.

$$Y''-Y'-2Y=9Ae^{3t}-3Ae^{3t}-2Ae^{3t}=4Ae^{3t}$$

Set it equal to $e^{3t} \Rightarrow 4A=1 \Rightarrow A=\frac{1}{4}$

Gen. sol'n: $y=C_1e^{2t}+C_2e^{-t}+\frac{1}{4}e^{3t}$.

Remark: The exponential coefficient of $Y(t)$ should be consistent to that of $g(t)$.

Example: $y'' - 2y' + y = 3\sin 3t$

Idea: RHS is a trig. func. Try trig. funcs.

Comp. sol'n: $r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0, \Rightarrow r = 1, 1$

$$y_c = C_1 e^t + C_2 t e^t.$$

$Y = A\cos 3t + B\sin 3t.$ Both cos & sin should appear.

$Y' = -3A\sin 3t + 3B\cos 3t,$

$Y'' = -9A\cos 3t - 9B\sin 3t.$ $(\sin t)' = \cos t, (\cos t)' = -\sin t.$

$$Y'' - 2Y' + Y = \underline{-9A\cos 3t} - 9B\sin 3t + 6A\sin 3t - \underline{6B\cos 3t} + \underline{A\cos 3t} + B\sin 3t$$

$$= (-9A - 6B + A)\cos 3t + (-9B + 6A + B)\sin 3t$$

$\Rightarrow -8A - 6B = 0, -8B + 6A = 3 \Rightarrow B = -\frac{4}{3}A, (-8)(-\frac{4}{3}A) + 6A = 3$

$\Rightarrow (\frac{32}{3} + 6)A = 3 \Rightarrow A = \frac{9}{32+18} = \frac{9}{50}. \quad B = -\frac{4}{3}A = -\frac{6}{25}$

$$Y = \frac{9}{50}\cos 3t - \frac{6}{25}\sin 3t$$

General solution: $y = C_1 e^t + C_2 t e^t + \frac{9}{50}\cos 3t - \frac{6}{25}\sin 3t.$

Rmk: Both cos and sin should appear.

Rmk: Only works for cos and sin. For other trig. funcs. the method fails.

Rmk: The trig. coefficients of Y should be consistent to that of $g(t)$.

Remark: Observe that derivatives of polynomials up to degree n is still a polynomial (up to degree n); derivatives of exp. funcs is still exp. funcs; derivatives of cosine & sine funcs is still cosine or sine funcs. This is why such method works. For functions without this property, this method fails. You have to use integration formula.

Rmk: polynomials, exp.s, cos & sin's are refered as basic types. Observe that product of basic type functions satisfy the same property.

We have seen that for

$$ay'' + by' + cy = g(t)$$

(i) If $g(t) = 3$ (or $g(t) = -10$), we should set $Y = A$

(ii) If $g(t) = t^2 + 1$ (or $g(t) = 6t^2$ or $g(t) = -2t^2 + 4t - 3$)

we should set $Y = At^2 + Bt + C$.

(iii) $g(t) = e^{3t}$, set $Y = Ae^{3t}$

(iv) $g(t) = 3\sin 3t$ (or $g(t) = 6\cos 3t$ or $g(t) = 18\cos 3t - 27\sin 3t$)

we should set $Y = A\cos 3t + B\sin 3t$

The list continues with products of basic types

(v) If $g(t) = 4t^3 e^{3t}$ (or $g(t) = (t^3 + 2t)e^{3t}$)

we should set $Y = (At^3 + Bt^2 + Ct + D)e^{3t}$.

Rmk: First deal with the polynomial. then supplement the exp. with the same exp. coefficient.

(vi) If $g(t) = (t^2+1)\sin 3t$ $\left(\begin{array}{l} \text{or } g(t) = 2t^2\cos 3t \\ \text{or } g(t) = 2t\cos 3t + (t^2+1)\sin 3t \end{array}\right)$

we should set $Y = (At^2 + Bt + C)\sin 3t + (Dt^2 + Et + F)\cos 3t$.

Rmk: First the polynomial, then one trig, then another. Keep the coefficient in the trigs. consistent for $Y(t)$

(vii) If $g(t) = e^{3t}\cos 4t$ (or $g(t) = -e^{3t}\sin 4t$, or $g(t) = 2e^{3t}\cos 4t - 6e^{3t}\sin 4t$)

we should set $Y = Ae^{3t}\cos 4t + Be^{3t}\sin 4t$

Rmk: First exp., then one trig, then another. exp. coeff's and trig. coeff's. should be consistent.

(viii) If $g(t) = 3te^{3t}\sin t$ $\left(\begin{array}{l}\text{or } g(t) = 6te^{3t}\cos t \\ \text{or } g(t) = 18te^{3t}\cos t - 7e^{3t}\sin t\end{array}\right)$

we should set: $Y = (At+B)e^{3t}\cos t + (Ct+D)e^{3t}\sin t$

Rmk: First deal with polynomial, then exponents, then one trig, then another trig. Keep the exponent coefficients and trig. coeff. consistent.

Example: $y'' + 4y = te^{t}\cos 2t.$

$(fg)' = f'g + fg'$

$(fgh)' = f'gh + fg'h + fgh'$

Char. eqn. $r^2 + 4 = 0 \Rightarrow r = \pm 2i$

Comp. sol'n: $y_c = C_1\cos 2t + C_2\sin 2t$

Set $Y = (At+B)e^{t}\cos 2t + (Ct+D)e^{t}\sin 2t$

$$\begin{aligned} Y' &= \underline{Ae^{t}\cos 2t} + \underline{(At+B)e^{t}\cos 2t} + (At+B)e^{t}(-2\sin 2t) \\ &\quad Ce^{t}\sin 2t + (Ct+D)e^{t}\sin 2t + \underline{(Ct+D)e^{t}(2\cos 2t)} \\ &= ((A+2C)t + A + B + 2D)e^{t}\cos 2t + ((-2A+C)t + C + D - 2B)e^{t}\sin 2t. \end{aligned}$$

$$Y'' = (A+2C)e^t\cos 2t + \left((A+2C)t + A+B+2D\right)\left(e^t\cos 2t\right)'$$
$$+(-2A+C)e^t\sin 2t + \left((-2A+C)t + C+D-2B\right)\left(e^t\sin 2t\right)'$$
$$= (A+2C)\underline{e^t\cos 2t} + \left((A+2C)t+A+B+2D\right)\left(\underline{e^t\cos 2t} - 2e^t\sin 2t\right)$$
$$+(-2A+C)e^t\sin 2t + \left((-2A+C)t + C+D-2B\right)\left(e^t\sin 2t + 2\underline{e^t\cos 2t}\right)$$
$$= \left(A+2C+(A+2C)t+A+B+2D+(-4A+2C)t+2C+2D-4B\right)e^t\cos 2t$$
$$\left((-2A-4C)t-2A-2B-4D-2A+C+(-2A+C)t+C+D-2B\right)e^t\sin 2t$$
$$= \left((-3A+4C)t+2A-3B+4C+4D\right)e^t\cos 2t$$
$$\left((-4A-3C)t\ -4A-4B+2C-3D\right)e^t\sin 2t.$$

$$4Y = (4At+4B)e^t\cos 2t + (4Ct+4D)e^t\sin 2t$$

$$Y''+4Y = \left((A+4C)t+2A+B+4C+4D\right)e^t\cos 2t$$
$$\left((-4A+C)t-4A-4B+2C+D\right)e^t\sin 2t$$

Set it equal to $te^t\cos 2t$

$$\Rightarrow A+4C=1.\quad 2A+B+4C+4D=0$$
$$-4A+C=0,\quad -4A-4B+2C+D=0$$
$$\Rightarrow C=4A \Rightarrow 17A=1 \Rightarrow A=\frac{1}{17},\quad C=\frac{4}{17}.$$
$$\Rightarrow B+4D=-2A-4C=\frac{-18}{17}$$
$$-4B+D=4A-2C=\frac{4}{17}-\frac{8}{17}=-\frac{4}{17}$$

In class I wrote the 4 as 2 mistakenly.

$$\Rightarrow B = \frac{-18}{17} - 4D \Rightarrow \frac{72}{17} + 16D + D = -\frac{4}{17} \Rightarrow 17D = \frac{-76}{17}$$

$$\Rightarrow D = -\frac{76}{289} \Rightarrow B = \frac{-18\times 17}{17\times 17} + \frac{4\times 76}{289} = \frac{-306+304}{289} = \frac{-2}{289}$$

$$Y = \left(\frac{1}{17}t - \frac{2}{289}\right)e^t \cos 2t + \left(\frac{4}{17}t - \frac{76}{289}\right)e^t \sin 2t$$

$$y = C_1 \cos 2t + C_2 \sin 2t + \left(\frac{1}{17}t - \frac{2}{153}\right)e^t \cos 2t + \left(\frac{4}{17}t - \frac{40}{153}\right)e^t \sin 2t$$

(ix) If $g(t) = e^{3t} + \sin 2t + t^2 e^t \cos 3t$ or any sums with different exp. coeff.s and trig. coeffs., we deal with each summand independently.

Example: $y'' + y = e^t + 2e^{3t} + \cos 2t + t + 4$.

Comp. sol'n: $y_c = C_1 \cos t + C_2 \sin t$

Set Y_1 to be a sol'n to $y'' + y = e^t$

$$Y_1 = Ae^t. \Rightarrow Y_1'' + Y_1 = 2Ae^t = e^t \Rightarrow A = \frac{1}{2} \Rightarrow Y_1 = \frac{1}{2}e^t$$

Set Y_2 to be a sol'n to $y'' + y = 2e^{3t}$

$$Y_2 = Be^{3t} \Rightarrow Y_2'' + Y_2 = 10Be^{3t} = 2e^{3t} \Rightarrow B = \frac{1}{5} \Rightarrow Y_2 = \frac{1}{5}e^{3t}$$

Set Y_3 to a sol'n to $y'' + y = \cos 2t$

$$Y_3 = C\cos 2t + D\sin 2t \Rightarrow Y_2'' + Y_2 = -3C\cos 2t - 3D\sin 2t = \cos 2t$$

$$\Rightarrow C = -\frac{1}{3}, D = 0 \Rightarrow Y_3 = -\frac{1}{3}\cos 2t$$

Set Y_4 to be a sol'n to $y'' + y = t + 4$

$$Y_4 = Et + F \Rightarrow Y_4'' + Y_4 = Et + F = t + 4 \Rightarrow E = 1, F = 4$$

$$\Rightarrow Y_4 = t + 4$$

By Principle of Superposition. $Y = Y_1 + Y_2 + Y_3 + Y_4$ is a sol'n to the original ODE

Gen. sol'n $y = C_1 \cos t + C_2 \sin t + \frac{1}{2} e^t + \frac{1}{5} e^{3t} - \frac{1}{3} \cos 2t + t + 4$

Attendance Quiz: $y'' - 5y' - 6y = te^{2t}$. Find the general sol'n.

HW 14: Skip #2 and #3b.

$$y_c = C_1 e^{6t} + C_2 e^{-t}$$

$$Y = (At + B)e^{2t}, \quad Y' = A \cdot e^{2t} + 2(At + B)e^{2t} = (2At + 2B + A)e^{2t}$$

$$Y'' = 2Ae^{2t} + (2At + 2B + A) \cdot 2e^{2t} = (4At + 4B + 4A)e^{2t}$$

$$Y'' - 5Y' - 6Y = (\underline{4At} + 4B + 4A - \underline{10At} - 10B - 5A - \underline{6At} - 6B)e^{2t}$$

$$= (-12At - A - 12B)e^{2t} = te^{2t}$$

$$-12A = 1, \quad -A - 12B = 0 \Rightarrow A = -\frac{1}{12}, \quad B = -\frac{1}{12} A = \frac{1}{144}.$$

$$y = C_1 e^{6t} + C_2 e^{-t} + \left(-\frac{1}{12} t + \frac{1}{144}\right) e^{2t}.$$

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